Credit Risk modelling

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1 Introduction

Credit risk is the risk of financial loss due to a debtor's default on a loan. The risk emanates from both actual and perceived defaults. Central to credit risk is the modelling of the default event. There has been a lot of academic interest in modelling credit risk and consequently there exists a vast body of literature. Moreover the credit crisis of 2008 sparked a renewed interest amongst market practitioners and regulators.

There exist two distinctly parallel worlds in credit risk mainly due to the dichotomy of data availability in the financial markets:

1. direct measurements of credit performance such as downgrades and defaults and

2. credit performance implied by corporate bond prices.

This distinct form of data focus has led to two distinct streams of data modelling. One is the world of physical credit defaults whereas the other is the world of implied default risk. Credit rating agencies such as Moody's and credit scoring agencies in consumer lending have traditionally been more interested in the world of actual (historical) credit defaults. However, the bulk of academic literature has focused on the implied side which naturally requires some form of modelling to link corporate bond prices and credit risk.

This paper reviews the development in credit risk models. The review is not exhaustive in any sense given the vastness of the literature available. The next section discusses the main types of credit risk models that exist. We then have a section on the basics of bond pricing and follow up with a more detailed discussion about the main types of credit risk models. We finish off with a brief discussion of dependence modelling in credit risk which is required when dealing with credit derivatives which have exposures to multiple names in a credit portfolio.

2 Introduction to credit risk models

There are two main types of credit risk models:

1. Structural models and

2. Intensity based models.

We start off with a discussion of structural models which preceded intensity based models. The bulk of this paper deals with intensity based models. We also set up a general calibration approach for intensity based models. In the course of the review we briefly discuss the issue of recovery in the event of credit default.

The earliest structural form models are based on Merton (1974) model. In this approach, a firm's default depends on the value of the assets it holds. This implies that default occurs when the market value of its assets is lower than the value of its liabilities. The payment to the debtors at the maturity of debt is therefore the smaller of the face value of the debt or the market value of the firm's assets. This basic intuition led to Merton's formula.

In addition to Merton (1974), Black and Cox (1976), Geske (1977), and Vasicek (1984) also developed some of the earlier structural form models. These models try to improve the original Merton framework by relaxing one or more of the unrealistic assumptions. Despite the improvements these models have limitations when it comes to practical application. One reason for this is due to the unrealistic assumption that defaults happen at the maturity of the debt as in the original Merton framework. Another simplifying but unrealistic assumption is that the risk free term structure is flat.

To overcome the drawbacks mentioned above, more sophisticated structural form models were developed which assume that a firm may default any time between the issuance and maturity of the debt. Short-term interest rates were modelled stochastically. In this scenario, the default occurs whenever the market value of the firm breaches a predetermined barrier. These structural form models include Kim, Ramaswamy and Sundaresan (1993) and Longstaff and Schwartz (1995). However, while these later structural form models provided some improvements over the earlier models, they still perform poorly in empirical analysis. For example the firm's market value still needs to be estimated, however not all the firm's assets have observable market values. In addition, these models do not take into account credit rating changes for risky corporate debt of the firms.

The intensity based models, or reduced form models as they are also known avoid these two problems by design. In these models, the time of bankruptcy is modelled as an exogenous event. This approach is advantageous for two reasons. First, it allows exogenous assumptions to be imposed only on observables. Second, it can easily be modified to include credit rating and can thus be used to price more sophisticated payouts.

Credit ratings are important because they also allow one to make objective estimates about the financial health of the firm without requiring information about its market value. Jarrow, Lando and Turnbull (1997) studied the term structure of credit risk spreads in a model with credit ratings. By incorporating the credit ratings, Duffie and Singleton (1999) presented a new approach for modeling the valuation of contingent claims subject to default and focussed on the term structure of interest rates for corporate bonds. Their study differed from other reduced form models by the way they parameterized the losses in case of default. Last but not least, Duffie (1999) discussed the empirical performance of the reduced form models and showed that these models may not be useful in explaining the relatively at or steeper yields for firms with low credit or higher credit risks, respectively.

3 Bond pricing basics

In this section we lay out the basic bond pricing set up that we will be used as a reference for the rest of this paper. We consider a probablility space (Ω, F, Q) equipped with the filtration $(F_t)_{t\geq 0}$ satisfying the usual conditions of right continuity and completeness.

1. Q is the risk neutral probability measure. Unless stated otherwise the modelling is carried out under this measure

2. $r(t)$ is the short interest rate.

3. Time to default is τ and survival indicator $I(t) := 1_{\tau>t}$.

4. Default free zero coupon bond (ZCB) prices for all maturities $T \geq t$ are $B(t, T)$

5. Defaultable zero coupon bond (DZCB) prices for all maturities $T \geq t$ are $\bar{B}(t, T)$

6. Absence of arbitrage implies: $0 \leq \bar{B}(t, T) \leq B(t, T), \forall t \leq T$

7. Bond prices must be decreasing non negative functions of maturity with $\bar{B}(t, t) = B(t, t) = 1$.

8. $B(t, T_1) \geq B(t, T_2) > 0$ and $\bar{B}(t, T_1) \geq \bar{B}(t, T_2) > 0$ for all $t < T_1 < T_2$ and $\tau > t$

9. At time t all prices of ZCBs and DZCBs with maturities $T \geq t$ are known.

10. DZCBs have zero default recovery so

$$I(t)\bar{B}(t,T) = \begin{cases} \bar{B}(t,T), & \tau > t \\ \\ 0, & \tau \leq t \end{cases}$$

11. Under the Q measure default free interest rate dynamics are independent of default time.

12. Fundamental relationship under Q:

$$B(t,T) = E[e^{-\int_t^T r(s)ds}.1]$$

and

$$\bar{B}(t,T) = E[e^{-\int_t^T r(s)ds}.I(T)] = E[e^{-\int_t^T r(s)ds}].E[I(T)] = B(t,T)E[I(T)]$$

therefore,

$$\bar{B}(t,T) = B(t,T).P(t,T)$$

$P(t,T)$ is the implied probability of survival in $[t,T]$

Hence

$$P(t,T) = \frac{\bar{B}(t,T)}{B(t,T)}$$

and the implied probability of default is

$$P^{Def}(t,T) = 1 - P(t,T)$$

If $P(t,T)$ has a right sided derivative in T, the implied density of default time is

$$Q[\tau \in [T, T+dt|F_t] = -\frac{\partial}{\partial T}P(t,T)dt$$

Hence if we have ZCB and DZCB prices for all maturities we could construct a term structure of survival probabilities

13. Conditional survival probability is the probability of survival over $[T_1, T_2]$ as seen from t given no default at time T_1

$$P(t,T_1,T_2) = \frac{P(t,T_1)}{P(t,T_2)}, t \leq T_1 \leq T_2$$

14. Implied hazard rate is defined as conditional probability of default per unit time $\triangle t$ at time T as seen from tine $t < T$,

$$\frac{1}{\triangle t} P^{Def}(t, T, T + \triangle t) = \frac{1}{\triangle t}(1 - P(t, T, T + \triangle t))$$

Discrete implied hazard rate of default over $[T, T + \triangle t]$ as seen from t is

$$H(t, T, T + \triangle t) = \frac{1}{\triangle t} \frac{P^{Def}(t, T, T + \triangle t)}{P(t, T, T + \triangle t)} = \frac{1}{\triangle t}(\frac{P(t, T)}{P(t, T + \triangle t)} - 1)$$

Continuous implied hazard rate is defined as:

$$h(t, T) = \lim_{\triangle t \to 0} H(t, T, t + \triangle t) = -\frac{1}{P(t, T)} \frac{\partial}{\partial T} P(t, T)$$

15. Forward rate for default bonds are defined as:

$$\bar{F}(t, T_1, T_2) = \frac{\bar{B}(t, T_1)/\bar{B}(t, T_2) - 1}{T_2 - T_1}$$

whereas defaultable instantaneous continuously compounded forward rates for T as seen from t are

$$\bar{f}(t, T) = \lim_{\triangle t \to 0} \bar{F}(t, T, T + \triangle t) = -\frac{\partial}{\partial T} log \bar{B}(t, T)$$

16. The conditional probability of default per time interval $[T_1, T_2]$ is the spread of defaultable over default free forward rates discounted by the defaultable forward rate:

$$\frac{P^{Def}(t, T_1, T_2)}{T_1 - T_2} = \frac{\bar{F}(t, T_1, T_2) - F(t, T_1, T - 2)}{1 + (T_2 - T_1)(\bar{F}(t, T_1, T_2)}$$

$$= \frac{\bar{B}(t, T_2)}{B(t, T_1)}(\bar{F}(t, T_1, T_2) - F(t, T_1, T_2))$$

The discrete implied hazard rate of default is given by the spread of defaultable over default free forward rates discounted by the default free forward rate:

$$H(t, T_1, T_2) = \frac{\bar{F}(t, T_1, T_2) - F(t, T_1, T - 2)}{1 + (T_2 - T_1)(F(t, T_1, T_2))}$$

$$= \frac{B(t,T_2)}{B(t,T_1)}(\bar{F}(t,T_1,T_2) - F(t,T_1,T_2))$$

The (continuous) implied hazard rateof default at time $T > t$ as seen from time t is given by the spread of the defaultable over the default free contiuously compounded forwrd rates:

$$h(t,T) = \bar{f}(t,T) - f(t,T)$$

17. Recovery conventions: So far we have assumed that in the event of default, bond investors loose all their investment. However there is some kind of recovery on default given debt seniority. For modelling purposes the following recovery conventions can be considered:

    (a) Constant recovery (recovery of face value). Let $R \in [0,1]$ be the constant amount that is recoverable on default. Then the DZCB has value

    $$\bar{B}(t,T) = E[e^{-r(T-t)}(1_{\tau>T} + R(1_{\tau\leq T}))]$$
    $$= B(t,T) - B(t,T)(1-R)P[\tau \leq T]$$

    This is equal to the value of ZCB minus the value of the expected default loss

    (b) Equivalent recovery (recovery of an equivalent default free bond). Let $R \in [0,1]$ be the recovery at default of a constant fraction of an equivalent ZCB

4 Structural form models

In this section we look at structural form models in more detail. In credit risk modeling, this approach is also known as the firm value approach since a firm's default is driven by the value of its assets. It was inspired by the 1970s Black-Scholes-Merton methodology for financial option pricing. Two classic structural form models we discuss here are the Merton model (Merton, 1974) and the first-passage-time model (Black and Cox, 1976).

As mentioned earlier, the Merton model assumes that default occurs at the maturity of the debt if value of the firms assets are lower than the face value. Let D be the debt level with maturity date T, and let $V(t)$ be the value of the assets following a geometric Brownian motion:

$$dV(t) = \mu V(t)dt + \sigma V(t)dW(t)$$

with drift μ, volatility σ and the standard Wiener process $W(t)$. Using Ito's lemma it can be shown that,

$$\frac{V(t)}{V(0)} = exp(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t \sim Lognormal((\mu - \frac{1}{2}\sigma^2)t, \sigma^2 t)$$

Given this relationship we can evaluate the default probability $P(V(T) \leq D)$.

The notion of distance-to-default facilitates the computation of conditional default probability. Given the sample path of asset values up to t, one may first estimate the unknown parameters in geometric brownian by maximum likelihood method. According to Duffie and Singleton (2003), let the distance-to-default $X(t)$ be defined by the number of standard deviations such that $logV_t$ exceeds $logD$, i.e.

$$X(t) = (logV(t) - logD)/\sigma$$

X(t) is thus a drifted Wiener process of the form

$$X(t) = c + bt + W(t), t \geq 0$$

with

$$b = \frac{\mu - \sigma^2/2}{\sigma}$$

and

$$c = \frac{logV(0) - logD}{\sigma}$$

Consequently, the conditional probability of default at maturity date T is

$$P(V(T) \leq D|V(t) > D) = P(X(T) \leq 0|X(t) > 0) = \Phi(\frac{X(t) - b(T-t)}{\sqrt(T-t)})$$

where $\Phi(.)$ is the cumulative normal distribution function

The first-passage-time model by Black and Cox (1976) extends the Merton model so that the default event could occur as soon as the asset value reaches a pre-specified debt barrier. $V(t)$ hits the debt barrier once the distance-to-default process $X(t)$ hits zero. Given the initial distance-to-default $c = X(0) > 0$, the first-passage-time τ is defined as

$$\tau = inf[t \geq 0 : X(t) \leq 0]$$

τ follows the inverse Gaussian distribution. Schrodinger (1915) and Smoluchowski (1915) were the first to introduce the Inverse Gaussian distribution for first hitting time in a system where the particles follow a Brownian motion. Since then the

Inverse Gaussian has been studied extensively by many authors. Consequently, the probability density of τ here is given by

$$f(t) = \frac{c}{\sqrt{2\pi}} t^{-1.5} exp[-\frac{(c+bt)}{2t}], t \geq 0$$

The survival function $S(t)$ is defined by $P(\tau > t)$ for any $t \geq 0$ and is given by

$$S(t) = \Phi(\frac{c+bt}{\sqrt{t}}) - e^{-2bc}\Phi(\frac{-c+bt}{\sqrt{t}})$$

The hazard rate $\lambda(t)$, also called the conditional default rate, is defined by the instantaneous rate of default conditional on the survivorship,

$$\lambda(t) = \lim_{\triangle t \to 0} \frac{1}{\triangle t} P[t < \tau < t + \triangle t | \tau \geq t)] = \frac{f(t)}{S(t)}$$

Using the inverse Gaussian density and survival functions, the form of the first-passage-time hazard rate is:

$$\lambda(t, c, b) = \frac{\frac{c}{\sqrt{2\pi t^3}} exp[-\frac{(c+bt)^2}{2t}]}{\Phi(\frac{c+bt}{\sqrt{t}}) - e^{-2bc}\Phi(\frac{-c+bt}{\sqrt{t}})}$$

This is one of the most important forms of hazard function in structural approach to credit risk modeling.

5 Intensity Models

Structural models are based on economic arguments and defaults are modelled as fundamental firm variables. Intensity based models, also referred to as reduced form models do not take fundamental economic arguments into account but model defaults as exogenously driven events. For example one type of model assumes defaults follow a Poisson process. This leads to better tractability and some claim a better empirical performance.

5.1 Poisson process

Let $N(t)$ be a Poisson process with intensity λ. It is hence an increasing process in the integers 0,1,.....,n, where $T_1, ..., T_n$ denote the jump times in the process. The process increments $N(T_{i+1}) - N(T_i)$ are indepedent and

$$P[N(T) - N(t) = n] = \frac{\lambda^n}{n!}(T - t)^n exp[-\lambda(T - t)]$$

for all $0 \leq t \leq T$

In the intensity based approach the default time is set equal to the first jump time of the Poisson process N. Thus $\tau = T_1$ is exponentially distributed with intensity parameter λ and the default probability is given by

$$F(t) = P[\tau \leq t] = 1 - e^{-\lambda t}$$

The intensity is the conditional default arrival rate given no default:

$$\lim_{h \to 0} \frac{1}{h} P[\tau \in (t, t + h)] | \tau > t] = \lambda$$

Let f denote the density of F hence

$$\lambda = \frac{f(t)}{1 - F(t)}$$

Survival probability for the Poisson process is

$$P^{surv}(0, T) = e^{-\lambda T}$$

The hazard rate:

$$H(t, T, T + \triangle t) = \frac{1}{\triangle t}(e^{\lambda \triangle t} - 1),$$

$$h(t, T) = \lambda$$

The hazard rate which is equal to the spread of the DZCB over the ZCB for the relevant term does not depend on time here. This means that the term structure of credit spreads in flat for the Poisson process case. This is clearly unrealistic given the empirical evidence and hence we need more sophisticated intensity models to model the term structure of credit spreads.

5.2 Inhomogenous Poisson process

The simple Poisson process N can be generalised to have time varying intensities $\lambda(t)$. The process increments $N(T_{i+1}) - N(T_i)$ are indepedent and

$$P[N(T) - N(t) = n] = \frac{1}{n!}(\int_t^T \lambda(s) ds)^n exp[-\int_t^T \lambda(s) ds]$$

The probability of default is then given by

$$P[\tau \leq t] = 1 - P[N(t) = 0] = 1 - e^{-\int_0^t \lambda(s)ds}$$

For the inhomogenous Poisson process,

$$\bar{B}(t,T) = B(t,T)e^{-\int_t^T \lambda(s)ds}$$

so we can fit a term structure of defaultable bond prices. The continuously compounded yield spread of this bond over the equivalent defalt free bond is

$$\frac{1}{T-t}\int_t^T \lambda(s)ds$$

Examples of how λ can be defined:

1. Constant: $\lambda(t) = \lambda$ for all t (the homogeneous Poisson case)
2. Linear: $\lambda(t) = a + bt$
3. Piece-wise constant: $\lambda(t) = a_1 + a_2 1_{t \geq t_1} + a_3 1_{t \geq t_2} + ...$ The parameters must be chosen such that $\lambda(t) \geq 0$ for all t.

5.2.1 Calibration setup

In this section we set up a calibration example where the Intensity λ is bootstrapped. This set up is quite generic and is based on the discussion by Giesecke (2002)

Assuming constant rate $r > 0$ and zero recovery rate for a DZCB maturing at T we get

$$\bar{B}(t.T) = E[e^{-\int_t^T r(s)ds}.I(T)] = E[e^{-\int_t^T r(s)ds}].E[I(T)] = B(t.T)E[I(T)] = e^{-(r+\lambda)(T-t)}$$

In order to make the modelling more realistic we calibrate a piece-wise constant intensity model to market prices of defaultable bonds hence

$$\lambda(t) = a_1 + a_2 1_{t \geq t_1} + a_3 1_{t \geq t_2} + ...$$

Suppose we have similary rated (for example AA rated) DZCBs with maturities $T_1 < T_2 < .... < T_n$ with respective price quotes $Q_1; Q_2; ......; Q_n$. We thus have have

$$Q_i e^{rT_i} = P[\tau > T_i] = e^{-\int_0^{T_i} \lambda(s)ds}$$

The coefficients can thus be bootstrapped and we can estimate a_1 from Q_1, a_2 from Q_2, and so on. For example suppose $n = 3$ and $T_i = i$ years. We thus have the following calibration setup

$$Q_1 e^r = e^{-\int_0^1 a_1 ds} = e^{-a_1}$$

$$Q_2 e^{2r} = e^{-\int_0^2 (a_1 + a_2 1_{s \geq 1}) ds} = e^{-2a_1 - a_2}$$

$$Q_3 e^{3r} = e^{-\int_0^3 (a_1 + a_2 1_{s \geq 1}) + a_3 1_{s \geq 2}) ds} = e^{-2a_1 - a_2}$$

allowing us to compute first $a1$, then $a2$, and afterwards $a3$. Now the piece-wise constant intensity model is fully calibrated to AA rated DZCB prices.

Since λ for the inhomogenous case depends only on time this is *not stochastic*. To incorporate stochasticity in spreads we look at implementing a Cox process to model the intensity. The Cox process is a generalisatiion of the Poisson process.

5.3 Cox process

A Cox process N with intensity λ_t is a generalisation of the inhomogenous Poisson process where the intensity is a random process, with the restriction that conditional on the realisation of λ, N is an inhomogenous Poisson process λ can be defined as :

$$d\lambda(t) = \mu_\lambda(t)dt + \sigma_\lambda(t)dW(t)$$

where $W(t)$ is the standard Brownian motion. The conditional and unconditional default probabilities are given by

$$P[\tau \leq t|\lambda] = 1 - P[N(t) = 0|\lambda] = 1 - e^{-\int_0^t \lambda_s ds}$$

$$P[\tau \leq t] = E[P[\tau \leq t|\lambda]] = 1 - E[e^{-\int_0^t \lambda_s ds}]$$

In finance, stochastic intensity based models are mostly the term-structure models borrowed from the literature of interestrate modeling. An obvious choice is to model the (stochastic) intensity using the one factor Cox-Ingersoll-Ross (CIR) model which is a *positive* mean reverting model. λ is defined as follows under CIR

$$d\lambda(t) = \kappa(\theta - \lambda(t))dt + \sigma\sqrt{\lambda(t)}dW_t$$

$\lambda(t)$ has a non-central chi-squared distribution where the probability density $f(\lambda))$ is defined as:

$$f(\lambda) = \frac{f_{\chi^2}(\frac{\lambda}{s}, \nu, \delta)}{s}$$

where ν is the known degrees of freedom, δ is the non centrality parameter and s

is a scalar and:

$$\nu = \frac{4\theta\kappa}{\sigma^2}$$

$$\delta = \frac{4\kappa e^{\kappa t}\lambda(0)}{\sigma^2(1 - e^{-\kappa t}}$$

$$s = \frac{\sigma^2(1 - e^{-\kappa t})}{4\kappa}$$

The mean and variance are:

$$E[\lambda(t)] = \lambda(0)e^{-\kappa t} + \theta(1 - e^{-\kappa t})$$

$$Var[\lambda(t)] = \lambda(0)\frac{\sigma^2}{\kappa}(e^{-\kappa t} - e^{-2\kappa t}) + \theta\frac{\sigma^2}{2\kappa}(1 - e^{-\kappa t})^2$$

The term-structure models provide straightforward ways to simulate the future default intensity for the purpose of predicting the conditional default probability. However, the models lack fundamental interpretation of the default events. Thus default intensities deserve more flexible and meaningful forms and one way is to allow for dependence of default intensity on state variables $z(t)$ (e.g. macroeconomic covariates) through a multivariate term-structure model for the joint distribution of $(\lambda(t), z(t))$. This approach essentially presumes a linear dependence in the diffusion components, e.g. by correlated Wiener processes.

6 Dependence modelling

The issue of default correlation embedded in credit portfolios has drawn intense discussions in the recent credit risk literature, in particular given the recent credit crisis. Among other works, Das, et al. (2007) performed an empirical analysis of default times for U.S. corporations and provided evidence for the importance of default correlation. Default correlation can be effectively characterized by multivariate survival analysis. There are broadly two different approaches to model dependence:

1. correlating the default intensities through the common covariates,

2. correlating the default times through the copulas.

The first approach is better known as the conditionally independent intensity based approach, in the sense that the default times are independent conditional on the common covariates. Examples of common covariates include the market-wide variables, e.g. the GDP growth, the short-term interest rate.

The copula approach to default correlation considers the default times as the modeling basis. A copula $C : [0,1]^n \rightarrow [0,1]$ is a function that is used to formulate the multivariate joint distribution based on the marginal distributions. By Sklar's theorem, for a multivariate joint distribution, there always exists a copula that can link the joint distribution to its univariate marginals. Therefore, the joint survival distribution of$(\tau_1, ..., \tau_n)$ can be characterized by $S_{joint}(t_1, ..., t_n) = C(S_1(t_1), ..., S_n(t_n))$, upon an appropriate selection of copula C. There is more detail on copulas in the appendix. In practice when modeling credit portfolios, it is important to check the appropriateness of the assumption behind the copula approach. The collapse of the CDO market was an obvious example of an oversimplification of copula modelling with reliance on the simple Gaussian copula to model dependence among multiple essentially heterogeneous credit exposures.

7 Conclusion

The structural approach is economically sound, however, it implies empirically less plausible spreads. The intensity based approach is adhoc, tractable, and empirically plausible. Structural and intensity based approach are not consistent as in the usual structural approach an intensity does not exist (this is due to the predicability of defaults). By introducing incomplete information in a structural model, both approaches can be unified to an extent. This will provide some economic underpinnings for the ad hoc nature of the intensity based framework.

8 Appendix: Copulas

Dependence between random variables is indicated by their joint distributions. Correlations measure linear dependencies, and thus correlations are not effective when working with nonlinear dependencies. Thus there is need of knowledge of the complete joint distribution. However, deriving joint distributions are generally next to impossible without heavy assumptions. At best one can hope to approximate marginal distributions.

Copulas are functions for *coupling* the marginal distributions to joint distributions. Traditional multivariate analysis combines dependence and joint distribution, whereas copulas separate dependence and marginal distributions. We assume that

$$F_1(x) = P[X \leq x]$$

and

$$F_2(y) = P[Y \leq y]$$

are cumulative distribution functions (CDFs) of the random variables X and Y and

$$F(x, y) = P[X \leq x, Y \leq y]$$

is their joint distribution. According to the theory under fairly general conditions there is a unique function C called a copula, such that

$$F(x, y) = C(F_1(x), F_2(y))$$

Thus if we know C the joint distribution $F(x, y)$ can be derived from the marginal distributions $F_1(x)$ and $F_2(x)$. Denoting the probabilities $s = F_1(x)$ and $t = F_2(y)$, we can (usually) take $x = F_1^{-1}(s)$ and $y = F_2^{-1}(t)$. Then, by Sklar's theorem we have

$$F(x, y) = F(F_1^{-1}(s), F_2^{-1}(t)) = C(s, t)$$

In essence if we have the marginal distributions and the copula we can get to the joint distribution. Common examples of copulas are:

1. Gaussian: $C_\Sigma(x_1, x_2, ..., x_n) = \Phi_\Sigma(\Phi^{-1}(x_1), ..., \Phi^{-1}(x_n))$
2. Student-t: $C_{\Sigma,\nu}(x_1, x_2, ..., x_n) = \Theta_{\Sigma,\nu}(\Theta_\nu^{-1}(x_1, ..., x_n))$
3. Archimedean: $C_\Psi(x_1, x_2, ..., x_n) = \Psi^{-1}(\Sigma_{i=1}^n \Psi(x_i))$

where $\Sigma \in R^{nxn}, \Phi_\Sigma$ denotes the multivariate normal distribution, $\Theta_{\Sigma,\nu}$ denotes multivariate Student-t distribution with degrees of freedom ν, and Ψ is the generator of Archimedean copulas

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